Scaling limit of the one-dimensional attractive Hubbard model: The non-half-filled band case

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Abstract

The scaling limit of the less than half filled attractive Hubbard chain is studied. This is a continuum limit in which the particle number per lattice site, n, is kept finite ($0 < n < 1$) while adjusting the interaction and bandwidth in a such way that there is a finite mass gap. We construct this limit both for the spectrum and the secular equations describing the excitations. We find, that similarly to the half filled case, the limiting model has a massive and a massless sector. The structure of the massive sector is closely analogous to that of the half filled band and consequently to the chiral invariant SU(2) Gross-Neveu (CGN) model. The structure of the massless sector differs from that of the half filled band case: the excitations are of particle and hole type, however they are not uniquely defined. The energy and the momentum of this sector exhibits a tower structure corresponding to a conformal field theory with $c = 1$ and SU(2)×SU(2) symmetry. The energy-momentum spectrum and the zero temperature free energy of the states with finite density coincides with that of the half filled case supporting the identification of the limiting model with the SU(2) symmetric CGN theory.

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I. INTRODUCTION

The one dimensional (1D) Hubbard model is described by the Hamiltonian

$$\hat{H} = -t \sum_{i=1}^{N} \sum_{\sigma=\uparrow,\downarrow} \left(c_{i,\sigma}^{+} c_{i+1,\sigma} + h.c. \right) + U \sum_{i=1}^{N} \left(\hat{n}_{i,\uparrow} - \frac{1}{2} \right) \left(\hat{n}_{i,\downarrow} - \frac{1}{2} \right)$$

$$+ \mu \sum_{i=1}^{N} \left(\hat{n}_{i,\uparrow} + \hat{n}_{i,\downarrow} \right) \tag{1.1}$$

in which $c_{i,\sigma}^{+}(c_{i,\sigma})$ creates (destroys) an electron at the site i with spin σ, $\hat{n}_{i,\sigma} = c_{i,\sigma}^{+} c_{i,\sigma}$. We impose periodic boundary conditions so the site $i = N + 1$ is the equivalent to the site $i = 1$. The hopping t is positive while the interaction U, for the considered attractive case, is negative, and the value of the chemical potential, μ, is choosen to fix the particle number per site, n. This model being completely integrable [1] and exactly solvable by the Bethe Ansatz (BA) [2] plays a central role in the theory of strongly correlated electron systems [3]. At the same time by direct linearization around the Fermi-points [4] both the *half-filled* ($n = 1$) and the *non-half-filled* ($n < 1$) Hubbard chains can be related to relativistic field theory models [5,6], specifically to the SU(2) symmetric chiral Gross-Neveu (CGN) model. Since the Hamiltonian of the CGN model has been diagonalized in the continuum (with a somewhat unorthodox cutoff i.e. filling up the Dirac sea up to a certain depth) by the BA method [7], constructing the relativistic (scaling) limit of the BA *solution* of the Hubbard chain can be of twofold interest. On the one hand it makes possible to study the relation between the two models in the detail, and on the other hand it promotes the Hubbard chain to an *integrable lattice* regularization of the CGN model.

Earlier Filev [8], later Melzer [9], recently Woynarovich and Forgács [10] have studied the relativistic limit of the *half-filled* (HF) Hubbard chain. In ref. [10] we have given a rather complete study of the structure of the states and the spectrum of the limiting model and also presented some convincing arguments that the scaling limit of the HF attractive Hubbard chain is the SU(2) (in fact SO(4)) symmetric CGN model. The scaling limit of the *non-half-filled* (NHF) attractive Hubbard chain has recently been studied by Woynarovich [11]. He has shown, that the limiting model possesses a massive sector whose spectrum and phaseshifts agree with those found in the scaling limit of the HF Hubbard chain.

The aim of the present work is to extend the study of Ref. [11] to the complete spectrum and to collect some other arguments supporting the equivalence of the scaling limit of the *non-half-filled* and that of the *half-filled* Hubbard chain. This also implies that the scaling limit of the *non-half-filled* Hubbard chain is the SO(4) symmetric CGN model.

The scaling limit is a continuum limit $N \to \infty$, $a \to 0$ so that $Na = L = $ const. (a and L being the lattice constant and the chain length, respectively), in which the particle number (N_e) per site $N_e/N \to n$ finite, i.e. $N_e \to \infty$ too. To avoid divergences, the interaction u has to be tuned in a special way (actually $u \to 0$). Finally, although a does not appear explicitly in $\hat{H}$, since $t \propto 1/\text{distance}$, $t \to \infty$ as $a \to 0$. All this is to be performed so that the gap in the spectrum of the unbound electrons is kept finite, that is [11]:

$$u \to 0, \quad t \to \infty; \quad N, N_e \to \infty, \quad \text{at} \quad N_e/N \to n = \text{const} < 1;$$
$$a \to 0 \quad \text{at} \quad Na = L = \text{const} \tag{1.2a}$$

so that

$$m_0 = \frac{8t}{\pi}\sqrt{u\sin^3(\pi n/2)}\exp\left\{-\frac{\pi\sin(\pi n/2)}{2u}\right\} = \text{const} \tag{1.2b}$$

and

$$2at\sin(\pi n/2) = 1 , \tag{1.2c}$$

We have performed this limit both in the spectrum and in the higher level Bethe Ansatz (HLBA) equations of the NHF Hubbard chain. We can summarise the properties of the limiting theory as follows:

(i) Like in the HF case, there are two kinds of excitations: massive and massless ones. While the massive sector is described in terms of well defined particles, in the massless sector the definition of the particless is not unique. (This is a major difference as compared to the HF case, where the excitations in both sectors are well defined particles.)

(ii) The massive particles have spin $1/2$. Their contribution to the energy and momentum is given as

$$\sum_j \epsilon(\kappa_j), \quad \text{and} \quad \sum_j p(\kappa_j), \tag{1.3a}$$

where

$$\epsilon(\kappa) = m_0\cosh(\kappa), \qquad p(\kappa) = m_0\sinh(\kappa), \tag{1.3b}$$

and κ_j's are the rapidities. The κ_j's and the set of variables χ_α describing the spin state of the particles, satisfy the following BA type equations

$$Lp(\kappa_j) = 2\pi I_j - \sum_{j'} \phi\left(\frac{\kappa_j - \kappa_{j'}}{\pi}\right) + \sum_\alpha 2\tan^{-1}\left(\frac{\kappa_j - \chi_\alpha}{\pi/2}\right) , \tag{1.4a}$$

$$\sum_j 2\tan^{-1}\left(\frac{\chi_\alpha - \kappa_j}{\pi/2}\right) = 2\pi J_\alpha + \sum_{\alpha'} 2\tan^{-1}\left(\frac{\chi_\alpha - \chi_{\alpha'}}{\pi}\right) , \tag{1.4b}$$

with

$$\phi(x) = \frac{1}{i}\ln\frac{\Gamma\left(\frac{1}{2} - i\frac{x}{2}\right)\Gamma\left(1 + i\frac{x}{2}\right)}{\Gamma\left(\frac{1}{2} + i\frac{x}{2}\right)\Gamma\left(1 - i\frac{x}{2}\right)} . \tag{1.5}$$

(1.4a,b) yield the same phaseshifts as those obtained for the half-filled band [9]

$$\psi^{tr} = \pi + \phi\left(\frac{\Delta\kappa}{\pi}\right), \quad \psi^s = \phi\left(\frac{\Delta\kappa}{\pi}\right) - 2\tan^{-1}\left(\frac{\Delta\kappa}{\pi}\right) \quad (\Delta\kappa = \kappa_j - \kappa_{j'}). \tag{1.6}$$

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(iii) The massless excitations correspond to states above resp. below a certain Fermi niveau (particles resp. holes). As the choice of the Fermi niveau is not unique, however, the parametrization of the states (number of particles and holes, and their momenta) is not uniquely defined. The energy and momentum, unlike in the HF case, consist not only of the sums of contributions of the individual excitations, but also contain certain 'collective' terms too. The spectrum of this sector shows the tower structure characteristic of a conformal field theory (CFT):

$$E - E_m = -\frac{\pi}{6L} + \frac{2\pi}{L}(x_{n,m} + \nu^+ + \nu^-)\,,$$
$$P - P_m = \frac{2\pi}{L}(s_{n,m} + \nu^+ - \nu^-)\,, \tag{1.7}$$

where E_m and P_m being the energy and momentum of the vacuum plus the massive sector, $\nu^\pm$ are integers and

$$x_{n,m} = \frac{1}{2}\left(n^2 + m^2\right)\,, \quad s_{n,m} = nm\,. \tag{1.8}$$

Here the numbers n and m are integers or half integers depending on the parity of the number of massive particles. In the case of an empty massive sector the apexes of the towers yield the conformal weights (the notations are those of [20])

$$\Delta = \frac{1}{4}(n+m)^2\,, \quad \bar{\Delta} = \frac{1}{4}(n-m)^2\,. \tag{1.9}$$

coinciding with those of a $c = 1$ CFT with an (enhanced) SU(2) symmetry.

(iv) The terms corresponding to the apexes of the towers in the spectrum of the massless sector depend nonlinearly on the number of the various excitations, hence we refer to them as collective terms. The parametrization of the massless sector (the Fermi niveau), however, can be choosen so, that the number of massive particles formally disappears from the collective terms and then the two sectors practically decouple.

(v) The energy-momentum spectrum of the limiting model is the same as that of the HF band case: the contribution of the massless sector as a function of the parameters m and n coincides with the corresponding contribution found in Ref. [10], and so does the contribution of the massive sector too, as the parity prescription for the I_j quantum numbers (i.e. the quantization of the momenta) expressed in terms of the m and n quantum numbers is also the same as that found in Ref. [10].

(vi) The ground state energy of states with a finite density of excitations is also in agreement with the result for the HF case. If we choose the Fermi surface so, that the two sectors decouple, the zero temperature free energies (which are nothing but the ground state energies in the presence of a chemical potential) of the different sectors become independent of each other, and the total free energy $f(\mu, \nu)$ can be written as

$$f(\mu, \nu) = -\frac{\mu^2}{\pi}\Psi\left(\frac{\mu}{m_0}\right) - \frac{\nu^2}{4\pi}\,, \tag{1.10}$$

where μ and ν denote the chemical potentials for the massive resp. massless particles. The $\Psi(\mu/m_0)$ can be given as an asymptotic series of $1/[\ln(\mu/m_0)]$ for high particle densities. (We remark that by parametrizing the massless sector in a different way, other terms depending nonlinearly on both chemical potentials would appear in Eq. (1.10).)

The equivalence of the limiting models of the HF and NHF cases is not trivial at all. While in a finite length chain we see a smooth behaviour as the bandfilling n is changed, in the thermodynamic limit the HF and NHF cases separate. There are three major (although not independent) differences.

– While the HF chain has two SU(2) symmetries (spin and isospin, this latter being connected to the charge), the NHF has only one.

– The structures of the excitation spectra are different, as in the HF case the gapless excitations are isospin $1/2$ particles, while in the NHF case these excitations are particle and hole type.

– In the (naive) continuum limit of the HF case there are umklapp processes violating chiral symmetry, while no such terms are present in the NHF case.

It is somewhat surprising, that some of these differences thought to be significant, disappear in the scaling limit: as we discussed in [10] there are indications, that the amplitude of the umklapp processes for the NHF chain scale out in a renormalization process, and as the conformal weights show, the SU(2) symmetry of the massless sector of the NHF chain develops in the scaling limit. The only significant difference we see after the scaling limit is in the structure of the massless sector. On the other hand, the fact, that the energy-momentum spectra of the two limiting theories coincide is a strong evidence supporting the equivalence of the two theories.

The paper is organized as follows. In Sec. II we give the BA equations of the Hubbard chain, and in Sec. III we describe the solutions corresponding to the less than half filled band. The scaling limit of the secular equations and the spectrum are constructed in Secs. IV and V respectively, and the results are discussed in Sec. VI together with a brief review of the states with a finite density of excitations, and we list the differences between the limiting models of the HF and NHF chains in Sec. VII. Appendix A we construct the naive continuum limit of the model.

II. THE BA EQUATIONS

The eigenvalue equations of the Hamiltonian (1.1) have been reduced to a set of nonlinear equations by Lieb and Wu [2]:

$$Nk_j = 2\pi I_j - \sum_{\alpha=1}^{M} 2\tan^{-1}\frac{\sin k_j - \lambda_\alpha}{U/4} \ , \tag{2.1a}$$

$$\left(I_j = \frac{M}{2}\,(\mathrm{mod}\,1)\right) ,$$

$$\sum_{j=1}^{N_e} 2 \tan^{-1} \frac{\lambda_\alpha - \sin k_j}{U/4} = 2\pi J_\alpha + \sum_{\beta=1}^{M} 2 \tan^{-1} \frac{\lambda_\alpha - \lambda_\beta}{U/2} \ . \tag{2.1b}$$

$$\left(J_\alpha = \frac{N_e + M + 1}{2} \ (\mathrm{mod}\,1) \right) .$$

Here N_e is the number of electrons, M is the number of down spins, i.e. $S^z = (N_e/2 - M)$, and the I_j and J_α quantum numbers are integers or half-odd-integers depending on the parities of N_e and M, as indicated. Once these equations are solved the wave-function can be given [12] and also the energy and the momentum of the corresponding state can be calculated:

$$E = NU/4 - \sum_{j=1}^{N_e} (2t \cos k_j + U/2 - \mu), \qquad P = \sum_{j=1}^{N_e} k_j \ . \tag{2.2}$$

For the considered $U < 0$ attractive chain near the ground-state most of the electrons form bound pairs with wavenumbers given (up to corrections exponentially small in N) as

$$\sin k^\pm = \Lambda \pm iu \tag{2.3}$$

with $u = |U|/4t$ and Λ being a subset of the set λ. By this relation $k^\pm$ can be eliminated from Eq.(2.1b) and one finds that the wavenumbers of the unbound electrons, the λs connected with their spin distribution and the Λs of the bound pairs satisfy the equations [13,14]

$$2\pi I_j = N k_j -$$
$$- \sum_{\alpha=1}^{n(\lambda)} \varphi_1(\sin k_j - \lambda_\alpha) - \sum_{\eta=1}^{n(\Lambda)} \varphi_1(\sin k_j - \Lambda_\eta) \ , \tag{2.4a}$$
$$\left(I_j = \frac{n(\lambda) + n(\Lambda)}{2} \ (\mathrm{mod}\,1) \right)$$

$$\sum_{j=1}^{n(k)} \varphi_1(\lambda_\alpha - \sin k_j) = 2\pi J_\alpha + \sum_{\beta=1}^{n(\lambda)} \varphi_2(\lambda_\alpha - \lambda_\beta) \ , \tag{2.4b}$$
$$\left(J_\alpha = \frac{n(k) + n(\lambda) + 1}{2} \ (\mathrm{mod}\,1) \right)$$

$$2\pi J_\eta = N \left(\sin^{-1}(\Lambda_\eta - iu) + \sin^{-1}(\Lambda_\eta + iu) \right) -$$
$$- \sum_{j=1}^{n(k)} \varphi_1(\Lambda_\eta - \sin k_j) - \sum_{\nu=1}^{n(\Lambda)} \varphi_2(\Lambda_\eta - \Lambda_\nu) \ . \tag{2.5}$$
$$\left(J_\eta = \frac{n(k) + n(\Lambda) + 1}{2} \ (\mathrm{mod}\,1) \right)$$

with

$$\varphi_m(\xi) = 2 \tan^{-1}(\xi/um) \tag{2.6}$$

Here $n(k)$, $n(\lambda)$ and $n(\Lambda)$ are the number of unbound electrons, the number of unbound electrons with down spins and the number of bound pairs, respectively ($N_e = n(k) + 2n(\Lambda)$, $M = n(\lambda) + n(\Lambda)$), and the quantum numbers I_j, J_α and J_η are integers or half-integers as indicated. (As in the following manipulations the quantum numbers may be redefined by absorbing certain constants into them, we give the 'parity prescriptions' for the quantum numbers together with the equations. All these prescriptions hold for even N) The energy and momentum expressed by these variables is

$$E = -Nut - \sum_j \left(2t(\cos k_j - u) - \mu\right) -$$

$$- \sum_\eta \left(2t\left(\sqrt{1 - (\Lambda_\eta - iu)^2} + \sqrt{1 - (\Lambda_\eta + iu)^2} - 2u\right) - 2\mu\right) , \qquad (2.7)$$

$$P = \frac{2\pi}{N}\left(\sum_j I_j - \sum_\alpha J_\alpha + \sum_\eta J_\eta\right) . \qquad (2.8)$$

III. STATES OF THE NON-HALF-FILLED HUBBARD CHAIN

1. Ground state – the physical vacuum

As a reference state consider the state, consisting of $N_r = n_r(\Lambda) < N/2$ bound pairs (and no k's, i.e. $N_e = 2N_r$), and the quantum number set J_η is given by

$$J_1 = -\frac{N_r - 1}{2}, \qquad J_{\eta+1} = J_\eta + 1, \qquad J_{N_r} = \frac{N_r - 1}{2}. \qquad (3.1)$$

If both $N, N_r \to \infty$ (at $N_r/N = n/2$ kept constant) the Λ_η's will be distributed in an interval $-B < \Lambda < B$ with a density σ_r that is defined so, that the number of Λ_ηs within the interval $(\Lambda, \Lambda + d\Lambda)$ is given as $N\sigma_r(\Lambda)d\Lambda$. It is not hard to see from (2.5), that this density is the derivative of the so called *counting function*:

$$\sigma_r(\Lambda) = \frac{dz_r(\Lambda)}{d\Lambda} , \qquad (3.2)$$

where

$$z_r(\Lambda) = \frac{1}{2\pi}\left\{\left(\sin^{-1}(\Lambda - iu) + \sin^{-1}(\Lambda + iu)\right) - \frac{1}{N}\sum_{\eta=1}^{N_r}\varphi_2(\Lambda - \Lambda_\eta)\right\} . \qquad (3.3)$$

This relation, applying the leading term of the Euler-Maclaurin type summation formula

$$\frac{1}{N}\sum_{\eta=1}^{n} f\left(\frac{J_\eta}{N}\right) = \int_{J_1/N - 1/2N}^{J_n/N + 1/2N} f(x)\,dx - \frac{1}{24N^2}\left(f'\left(\frac{J_n + 1/2}{N}\right) - f'\left(\frac{J_1 - 1/2}{N}\right)\right) \qquad (3.4)$$

leads to the equation

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$$\sigma_r(\Lambda) = \sigma_0(\Lambda) - \frac{1}{2\pi} \int_{-B}^{B} K_2(\Lambda - \Lambda')\sigma_r(\Lambda') \; , \tag{3.5}$$

$$\sigma_0(\Lambda) = \frac{1}{2\pi} 2\mathrm{Re}\left(\left(\sqrt{1 - (\Lambda - iu)^2} \right)^{-1} \right) \; , \tag{3.6}$$

$$K_m(\xi) = \frac{2mu}{(mu)^2 + \xi^2} \; , \tag{3.7}$$

where the limits are determined through

$$z_r(\pm B) = \pm N_r / 2N \tag{3.8a}$$

(i.e.

$$\int_{B}^{\infty} \sigma_r(\Lambda) = \int_{-\infty}^{-B} \sigma_r(\Lambda) = \frac{1}{2}\left(1 - \frac{2N_r}{N} \right) = \frac{1 - n}{2} \; , \tag{3.8b}$$

as it can be seen by integrating (3.5)).

The energy of this reference state is (using summation formula (3.4) in leading order)

$$\begin{aligned} E_r &= -utN + \sum \varepsilon_0(\Lambda_\eta) \\ &= -utN + N \int_{-B}^{+B} \varepsilon_0(\Lambda)\sigma_r(\Lambda) \; , \end{aligned} \tag{3.9}$$

with

$$\varepsilon_0(\Lambda) = -\left(4t\left(\mathrm{Re}\sqrt{1 - (\Lambda - iu)^2} - u \right) - 2\mu \right) \; . \tag{3.10}$$

In (3.9) the first term is irrelevant. The second term, iterating σ_r, can be transformed into the form

$$E_r + utN = N \int_{-B}^{+B} \varepsilon_r(\Lambda)\sigma_0(\Lambda) \; , \tag{3.11}$$

where ε_r satisfies the equation

$$\varepsilon_r(\Lambda) = \varepsilon_0(\Lambda) - \frac{1}{2\pi} \int_{-B}^{B} K_2(\Lambda - \Lambda')\varepsilon_r(\Lambda') \; . \tag{3.12}$$

It is easy to check, that the energy of (3.11) as a function of B, i.e. as a function of n, (at fixed μ) is minimal if

$$\varepsilon_r(B) = 0 \; . \tag{3.13}$$

As ε_0, depends on μ linearly so does ε_r, hence it is possible to choose a μ such that (3.13) holds at the desired bandfilling n. In the following we suppose that μ has this value, i.e. the reference state is the ground state.

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In general the (3.5) can not be solved analytically, and B as a function of n can not be given in a closed form. In the relativistic limit however, when $u \to 0$, (3.5) can be solved using the method devised by Yang and Yang [16]: the equations of the type (3.5) can be transformed into a series of Wiener-Hopf equations which can be solved with sufficient accuracy yielding [14]

$$B = \sin \frac{\pi n}{2} - \frac{u}{\pi} \left(1 + \ln \frac{\pi \cos^2 \frac{\pi n}{2} \sin \frac{\pi n}{2}}{2u} \right). \tag{3.14}$$

2. The possible excitations

The ground state can be excited by combinations of the following three "elementary" excitations:

i) introducing holes and particles in the Λ distribution by removing some J_η from the ground-state set and introducing some outside it,

ii) introducing complex Λs,

iii) introducing unbound electrons (real ks and λs).

The excitations of type i) with the proper choice of μ (described above) have a dispersion with no gap, while those of type ii) and iii) possess gaps. The scaling limit (1.2) is constructed so, that the gap in the spectrum of the excitations of type iii) is a fixed finite value (m_0). In this limit the gap of the excitations of type ii) diverges, i.e. they can be discarded. (In the following we shall refer to type i) excitations as particles and holes or massless excitations, and type iii) excitations will be called massive particles or unbound electrons.)

3. Particles and holes in the Λ set

First we consider the equations of the Λs (2.5). To describe the particle and hole type excitations properly we should define the 'Fermi sea'. If $n(k) + n(\Lambda) = n_r(\Lambda) \,(\text{mod}\, 2)$ the Fermi sea (of Js) is the ground state set of the Js, i.e. the Fermi sea is made up by all the J_f integers or half-integers satisfying

$$J_1 = -J_F + 1/2 , \qquad J_{f+1} = J_f + 1 , \qquad J_f \leq J_F - 1/2 , \tag{3.15}$$

with

$$J_F = N_r/2 . \tag{3.16}$$

If $n(k) + n(\Lambda) \neq n_r(\Lambda) \,(\text{mod}\, 2)$, there is no unique choice: as the parity of the J_ηs (i.e. that of the J_fs) is changed, keeping (3.15) we may chose any of $J_F = (N_r \pm 1)/2$. For the sake of definiteness we choose

$$J_F = (N_r + 1)/2 , \tag{3.17}$$

thus

$$J_F = \frac{N_r + \delta}{2}, \quad \delta = \begin{cases} 0 & \text{if } n(k) + n(\Lambda) = n_r(\Lambda) \,(\text{mod } 2) \\ 1 & \text{if } n(k) + n(\Lambda) \neq n_r(\Lambda) \,(\text{mod } 2) \end{cases}. \tag{3.18}$$

(We note that the Fermi sea defined above resembles most the ground-state distribution of the Λs, nevertheless many other definitions are possible: the only role of J_F is to provide a reference point in the space of the J quantum numbers connected to the Λ-rapidities, i.e. δ can be choosen even to depend explicitly on $n(k)$. We shall discuss this later.)

To proceed, we define the counting function:

$$z(\Lambda) = \frac{1}{2\pi} \left\{ \left(\sin^{-1}(\Lambda_\eta - iu) + \sin^{-1}(\Lambda + iu) \right) - \right.$$
$$\left. - \frac{1}{N} \sum_{j=1}^{n(k)} \varphi_1(\Lambda - \sin k_j) - \frac{1}{N} \sum_{\eta=1}^{n(\Lambda)} \varphi_2(\Lambda - \Lambda_\eta) \right\}, \tag{3.19}$$

in terms of which Eq. (2.5) reads

$$z(\Lambda_\eta) = \frac{J_\eta}{N}. \tag{3.20}$$

It is clear, that (3.20) has a solution for any $z(-\infty) < J < z(+\infty)$ replacing J_η. The rapidities of the particles, holes and respectively the elements of the Fermi sea are determined by the equations

$$z(\Lambda_p) = \frac{J_p}{N}, \qquad z(\Lambda_h) = \frac{J_h}{N}, \qquad z(\Lambda_f) = \frac{J_f}{N}, \tag{3.21}$$

respectively, where the J_ps are those J_ηs, which are not elements of the set J_f of (3.15), while the J_hs are those J_fs, which are not elements of the J_η set. (Note, that $\pm J_F$ themselvs can be neither holes nor particles, as their parity is different from that of the J_ηs and J_fs.) Comparing the parity prescription $J_\eta = (n(k) + n(\Lambda) + 1)/2$ and the number of elements in (3.15) we see, that the total number of ks (massive particles), particles and holes (massless excitations) is always even. (It has to be noted, that the above mentioned freedom in choosing J_F introduces an ambiguity in the *description* of certain states: depending on the choice of J_F, in the same state the number of massless excitations can be different. This ambiguity drops out, however, of physical quantities, like the energy and momentum which do not depend on the choice of J_F.) The particles and holes being defined, it is convenient to write $z(\Lambda)$ in the form

$$z(\Lambda) = \frac{1}{2\pi} \left\{ \left(\sin^{-1}(\Lambda_\eta - iu) + \sin^{-1}(\Lambda + iu) \right) - \right.$$
$$- \frac{1}{N} \sum_{j=1}^{n(k)} \varphi_1(\Lambda - \sin k_j) - \frac{1}{N} \sum_f \varphi_2(\Lambda - \Lambda_f)$$
$$\left. - \frac{1}{N} \sum_p \varphi_2(\Lambda - \Lambda_p) + \frac{1}{N} \sum_h \varphi_2(\Lambda - \Lambda_h) \right\}. \tag{3.22}$$

The density of the Λs is given by the derivative of the z:

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$$\sigma_t(\Lambda) = \frac{dz(\Lambda)}{d\Lambda}\,, \tag{3.23}$$

i.e.

$$\sigma_t(\Lambda) = \sigma_0(\Lambda) - \frac{1}{2\pi N}\sum_{j=1}^{n(k)} K_1(\Lambda - \sin k_j) - \frac{1}{2\pi N}\sum_f K_2(\Lambda - \Lambda_f)$$
$$- \frac{1}{2\pi N}\sum_p K_2(\Lambda - \Lambda_p) + \frac{1}{2\pi N}\sum_h K_2(\Lambda - \Lambda_h)\,. \tag{3.24}$$

Calculating the sum over the Fermi sea by means of (3.4) we arrive at

$$\sigma_t(\Lambda) = \sigma_0(\Lambda) - \frac{1}{2\pi N}\sum_{j=1}^{n(k)} K_1(\Lambda - \sin k_j)$$
$$- \frac{1}{2\pi N}\sum_p K_2(\Lambda - \Lambda_p) + \frac{1}{2\pi N}\sum_h K_2(\Lambda - \Lambda_h)$$
$$- \frac{1}{2\pi 24 N^2 \sigma_t(B^+)}K_2'(\Lambda - B^+) + \frac{1}{2\pi 24 N^2 \sigma_t(B^-)}K_2'(\Lambda - B^-)$$
$$- \int_{B^-}^{B^+} K_2(\Lambda - \Lambda')\sigma_t(\Lambda')\,. \tag{3.25}$$

where the limits B^+ and B^- defined by the equations

$$z(B^\pm) = \pm\frac{J_F}{N}\,, \tag{3.26a}$$

which are equivalent to

$$\int_{B^+}^{\infty} \sigma_t(\Lambda) = \int_{-\infty}^{B^-} \sigma_t(\lambda) = \frac{1}{2}\left\{1 - n - \frac{n(k) + n(p) - n(h) + 2\delta}{N}\right\}\,. \tag{3.26b}$$

As (3.25) is a linear equation, the density is a sum:

$$\sigma_t(\Lambda) = \sigma_b(\Lambda) + \frac{1}{N}\sigma_{\{k\}}(\Lambda) + \frac{1}{N}\sigma_{\{p\}}(\Lambda) + \frac{1}{N}\sigma_{\{h\}}(\Lambda) + \frac{1}{N^2}\sigma_{fsc}(\Lambda)\,, \tag{3.27}$$

where all terms (the contributions of the bulk, massive particles, the particles and holes, and the finite size corrections respectively) satisfy equations of the type

$$x(\Lambda) = \mathcal{I}_x(\Lambda) - \frac{1}{2\pi}\int_{B^-}^{B^+} K_2(\Lambda - \Lambda')x(\Lambda')\,, \tag{3.28}$$

where the inhomogeneous part, $\mathcal{I}_x(\Lambda)$, can be written as:

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$$\sigma_b(\Lambda): \quad \mathcal{I}_b(\Lambda) = \sigma_0(\Lambda)\,, \tag{3.29a}$$

$$\sigma_{\{k\}}(\Lambda): \quad \mathcal{I}_{\{k\}}(\Lambda) = -\frac{1}{2\pi}\sum_{j=1}^{n(k)} K_1(\Lambda - \sin k_j)\,, \tag{3.29b}$$

$$\sigma_{\{p\}}(\Lambda): \quad \mathcal{I}_{\{p\}}(\Lambda) = -\frac{1}{2\pi}\sum_{p} K_2(\Lambda - \Lambda_p)\,, \tag{3.29c}$$

$$\sigma_{\{h\}}(\Lambda): \quad \mathcal{I}_{\{h\}}(\Lambda) = +\frac{1}{2\pi}\sum_{h} K_2(\Lambda - \Lambda_h)\,, \tag{3.29d}$$

$$\sigma_{fsc}(\Lambda): \quad \mathcal{I}_{fsc}(\Lambda) = -\frac{1}{48\pi\sigma_t(B^+)}K_2'(\Lambda - B^+) + \frac{1}{48\pi\sigma_t(B^-)}K_2'(\Lambda - B^-)\,. \tag{3.29e}$$

The equations for the holes and particles can be given in terms of the density: noticing, that the particles and holes will condense in the vicinity of the Fermi points, based on (3.21), (3.26) and (3.23) we may write:

$$2\pi \int_{B^+}^{\Lambda_{p/h}^+} \sigma_t(\Lambda) = \frac{2\pi}{N}\Delta J_{p/h}^+ \quad \left(\Delta J_{p/h}^+ = J_{p/h}^+ - J_F = \text{half} - \text{integer}\right)\,,$$

$$2\pi \int_{\Lambda_{p/h}^-}^{B^-} \sigma_t(\Lambda) = \frac{2\pi}{N}\Delta J_{p/h}^- \quad \left(\Delta J_{p/h}^- = -J_F - J_{p/h}^- = \text{half} - \text{integer}\right)\,. \tag{3.30}$$

Here p/h is either p or h, and the upper index $+$ or $-$ refers to the side of Fermi sea to which the particle or hole is near, and the 2π factor is introduced for later convenience. Equations of the type (3.28) in general (unlike the half-filled-band case) can not be solved in a closed form, neither the Eqs. (3.30) can be given in a more explicit way, nevertheless they can be handled in the $u \to 0$ limit.

4. Unbound electrons

The unbound electrons are described by Eqs. (2.4) from which it is possible to eliminate the sum over the Λs using the summation formula (3.4) leading to

$$2\pi I_j = N k_j - \sum_{\alpha=1}^{n(\lambda)} \varphi_1(\sin k_j - \lambda_\alpha) -$$

$$- \sum_{p} \varphi_1(\sin k_j - \Lambda_p) + \sum_{h} \varphi_1(\sin k_j - \Lambda_h) -$$

$$- \frac{1}{24N\sigma_t(B^+)}K_1(\Lambda - B^+) + \frac{1}{24N\sigma_t(B^-)}K_1(\Lambda - B^-) -$$

$$- N\int_{B^-}^{B^+} \varphi_1(\Lambda - \Lambda')\sigma_t(\Lambda')\,. \tag{3.31}$$

The integral in (3.31) can be transformed in the following way: if a function satisfies (3.28), than it satisfies also the integral relation

$$\int_{B^-}^{B^+} \varphi_m(\xi - \Lambda)\, x(\Lambda) = -\left(\int_{-\infty}^{B^-} + \int_{B^+}^{\infty} \right) \varphi_m(\xi - \Lambda)\, x(\Lambda) +$$

$$+ \int_{-\infty}^{\infty} \varphi_m(\xi - \Lambda)\, \mathcal{I}_x(\Lambda) - \int_{B^-}^{B^+} \varphi_{m+2}(\xi - \Lambda)\, x(\Lambda)\,, \tag{3.32}$$

as it can be checked by calculating the convolution of φ_m with x of (3.28). It is easy to show by iteration, that this relation is equivalent to

$$\int_{B^-}^{B^+} \varphi_1(\xi - \Lambda)\, x(\Lambda) =$$

$$- \left(\int_{-\infty}^{B^-} + \int_{B^+}^{\infty} \right) 2\tan^{-1}\tanh \frac{\pi(\xi - \Lambda)}{4u} x(\Lambda) + \int_{-\infty}^{\infty} 2\tan^{-1}\tanh \frac{\pi(\xi - \Lambda)}{4u} \mathcal{I}_x(\Lambda)\,. \tag{3.33}$$

After substituting (3.33) into (3.31) and evaluating explicitly some of the integrals of the $\mathcal{I}(\Lambda)$ one arrives at

$$2\pi I_j = N\left\{ k_j - \int_{-\infty}^{\infty} 2\tan^{-1}\tanh \frac{\pi(\sin k_j - \Lambda)}{4u} \sigma_0(\Lambda) \right\} + \tag{3.34a}$$

$$+ N \left(\int_{-\infty}^{B^-} + \int_{B^+}^{\infty} \right) 2\tan^{-1}\tanh \frac{\pi(\sin k_j - \Lambda)}{4u} \sigma_t(\Lambda) - \tag{3.34b}$$

$$- \sum_{\alpha=1}^{n(\lambda)} 2\tan^{-1} \frac{\sin k_j - \lambda_\alpha}{4u} + \sum_{j'=1}^{n(k)} \phi\left(\frac{\sin k_j - \sin k_{j'}}{2u} \right) - \tag{3.34c}$$

$$- \sum_p 2\tan^{-1}\tanh \frac{\pi(\sin k_j - \Lambda_p)}{4u} + \sum_h 2\tan^{-1}\tanh \frac{\pi(\sin k_j - \Lambda_h)}{4u} - \tag{3.34d}$$

$$- \frac{1}{24N\sigma_t(B^+)} \frac{\pi}{2u} \frac{1}{\cosh \frac{\pi(\sin k_j - B^+)}{2u}} + \frac{1}{24N\sigma_t(B^-)} \frac{\pi}{2u} \frac{1}{\cosh \frac{\pi(\sin k_j - B^-)}{2u}}\,. \tag{3.34e}$$

For a general u we find this is the simplest form of the equations for the unbound electrons, but these equations further simplify in the scaling limit.

5. The energy and momentum

To calculate the energy, in (2.7) we evaluate the sum over the Λ by using again the Euler-Maclaurin formula (3.4):

$$E = -Nut - \sum_j \left(2t(\cos k_j - u) - \mu \right) + \sum_p \varepsilon_0(\Lambda_p) - \sum_h \varepsilon_0(\Lambda_h) -$$

$$- \frac{1}{24N\sigma_t(B^+)} \varepsilon_0'(B^+) + \frac{1}{24N\sigma_t(B^-)} \varepsilon_0'(B^-) + N \int_{B^-}^{B^+} \varepsilon_0(\Lambda)\sigma_t(\Lambda)\,. \tag{3.35}$$

Iterating σ_t in the integral by means of (3.24) one arrives at

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$$E = -Nut + N \int_{B^-}^{B^+} \varepsilon(\Lambda)\sigma_0(\Lambda) +$$

$$+ \sum_j \left(-(2t(\cos k_j - u) - \mu) - \frac{1}{2\pi} \int_{B^-}^{B^+} \varepsilon(\Lambda)K_1(\Lambda - \sin k_j) \right) +$$

$$+ \sum_p \varepsilon(\Lambda_p) - \sum_h \varepsilon(\Lambda_h) -$$

$$- \frac{1}{24N\sigma_t(B^+)}\varepsilon'(B^+) + \frac{1}{24N\sigma_t(B^-)}\varepsilon'(B^-), \tag{3.36}$$

where $\varepsilon(\Lambda)$ satisfies Eq. (3.28) with the inhomogeneous part:

$$\varepsilon(\Lambda): \quad \mathcal{I}_\varepsilon(\Lambda) = \varepsilon_0(\Lambda). \tag{3.37}$$

(Note, that $\varepsilon \neq \varepsilon_r$ as the limits in the integral equations (3.28) and (3.12) are different.) Next, to calculate the integral of $\varepsilon \times K_1$ we substitute the derivative of (3.33), and obtain

$$E = -Nut + N \int_{B^-}^{B^+} \varepsilon(\Lambda)\sigma_0(\Lambda) - \tag{3.38a}$$

$$- \left(\frac{1}{24N\sigma_t(B^+)}\varepsilon'(B^+) - \frac{1}{24N\sigma_t(B^-)}\varepsilon'(B^-) \right) + \tag{3.38b}$$

$$+ \sum_p \varepsilon(\Lambda_p) - \sum_h \varepsilon(\Lambda_h) + \tag{3.38c}$$

$$+ \sum_j \left(-(2t(\cos k_j - u) - \mu) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi}{2u} \frac{1}{\cosh \frac{\pi(\sin k_j - \Lambda)}{2u}} \varepsilon_0(\Lambda) \right) + \tag{3.38d}$$

$$+ \frac{1}{2\pi} \left(\int_{-\infty}^{B^-} + \int_{B^+}^{\infty} \right) \frac{\pi}{2u} \frac{1}{\cosh \frac{\pi(\sin k_j - \Lambda)}{2u}} \varepsilon(\Lambda). \tag{3.38e}$$

In this expression the first and second terms contain the bulk contributions, the third term is connected with the finite size corrections (but also the second term contains finite size corrections due to the deviation of $B^\pm$ from $\pm B$), and the rest is the excitation energy.

The momentum is given by (2.8), which after summation over the Fermi sea reads:

$$P = \frac{2\pi}{N} \left(\sum_j I_j - \sum_\alpha J_\alpha + \sum_p J_p - \sum_h J_h \right). \tag{3.39}$$

As in (3.30) the particle and hole rapidities are connected to $\Delta J_{p/h}^\pm = \pm J_{p/h}^\pm - J_F$ type expressions, and as J_F may have more values, we write

$$P = \sum_j \frac{2\pi I_j}{N} - \sum_\alpha \frac{2\pi J_\alpha}{N} + \sum_p^{(+)} \frac{2\pi \Delta J_p^+}{N} - \sum_h^{(+)} \frac{2\pi \Delta J_h^+}{N} -$$

$$- \sum_p^{(-)} \frac{2\pi \Delta J_p^-}{N} + \sum_h^{(-)} \frac{2\pi \Delta J_h^-}{N} + \frac{\pi \delta}{N}(\Delta n^+ - \Delta n^-) + \frac{\pi N_r}{N}(\Delta n^+ - \Delta n^-), \tag{3.40}$$

with

$$\Delta n^+ = n(p^+) - n(h^+), \quad \Delta n^- = n(p^-) - n(h^-)). \tag{3.41}$$

Here the $\sum^{(\pm)}$ means summations over the particles and holes near $\pm J_F$, with $n(p^\pm)$ resp. $n(h^\pm)$ being the numbers of the particles resp. holes near the Fermi points $\pm J_F$. As the last term can give an infinite contribution in the scaling limit (where the momentum must be devided by a), we have to redefine the lattice so, that this term be *equivalent* to zero. This can be done, if the bandfilling $n = 2N_r/N$ is a rational number. Suppose, that $n = l/q$, where l and q are relative prime numbers, and that ν and η are the smallest integers satifying $\nu = 4q\eta/l$. If in the redefined lattice ν lattice sites form one elementary cell, than the point $\pi N_r/N = \pi n/2$ of the original Brillouin zone will be transformed into the origin of the new Brillouin zone, so the last term of (3.40) can be dropped. (The redefinition of the lattice has consequences on the parities of some numbers: as the lattice must consist of an integer number of elementary cells, N and N_r must be integer multiples of 4 and 2, respectively)

IV. EQUATIONS OF THE EXCITATIONS IN THE SCALING LIMIT

1. The massless excitations

In Ref. [11] the spectrum and structure of the massive excitations have been investigated. It has been found, that in order to keep the mass gap (m_0) finite, relations (1.2) have to be obeyed. Now we examine the behaviour of the excitations connected with the Λ distribution in the limit (1.2).

First consider (3.30). As in the limit (1.2) $N \to \infty$, we divide these equations by a:

$$\frac{2\pi}{a} \int_{B^+}^{\Lambda^+_{p/h}} \sigma_t(\Lambda) = \frac{2\pi \Delta J^+_{p/h}}{L}, \qquad \frac{2\pi}{a} \int_{\Lambda^-_{p/h}}^{B^-} \sigma_t(\Lambda) = \frac{2\pi \Delta J^-_{p/h}}{L}. \tag{4.1}$$

As the r.h.s. is finite, in order to have the l.h.s. finite too,

$$\Lambda^\pm_{p/h} - B^\pm = O(a). \tag{4.2}$$

According to (1.2)

$$a = \frac{4}{\pi m_0}\sqrt{u\sin(\pi n/2)}\exp\left\{-\frac{\pi\sin(\pi n/2)}{2u}\right\}. \tag{4.3}$$

As the solutions of (3.28) change on the scale of u, the integrands in (3.30) are constants on a scale $\propto a$, and we may write:

$$2\pi\sigma_t(B^+)\frac{\Lambda^+_{p/h} - B^+}{a} = \frac{2\pi\Delta J^+_{p/h}}{L}, \qquad 2\pi\sigma_t(B^-)\frac{B^- - \Lambda^-_{p/h}}{a} = \frac{2\pi\Delta J^-_{p/h}}{L}. \tag{4.4}$$

Finally, as $\sigma_t(B^\pm) = \sigma_r(B) + O(N^{-1})$, in the scaling limit, one obtains:

$$2\pi\sigma_r(B)\frac{\Lambda^+_{p/h} - B^+}{a} = \frac{2\pi\Delta J^+_{p/h}}{L}, \qquad 2\pi\sigma_r(B)\frac{B^- - \Lambda^-_{p/h}}{a} = \frac{2\pi\Delta J^-_{p/h}}{L}. \tag{4.5}$$

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These equations can be considered as the secular equations of the massless excitations in the scaling limit. They look very similar to the secular equations of free particles, but they are not exactly the same, as B^+ and B^-, i.e. the location of the Fermi surface in the rapidity space depend on the number of excitations.

2. Massive particles

Now we consider the scaling limit of (3.34). The first term on the r.h.s. (line (3.34a)) can be evaluated to give

$$k_j - \int_{-\infty}^{\infty} 2 \tan^{-1} \tanh \frac{\pi(\sin k_j - \Lambda)}{4u} \sigma_0(\Lambda) = \frac{4}{\pi}\sqrt{u} \exp\left\{-\frac{\pi}{2u}\right\}. \tag{4.6}$$

This, in the limit (1.2) disappears for any $n < 1$.

In the second term (line (3.34b))

$$N\left(\int_{-\infty}^{B^-} + \int_{B^+}^{\infty}\right) 2 \tan^{-1} \tanh \frac{\pi(\sin k_j - \Lambda)}{4u} \sigma_t(\Lambda) =$$

$$\frac{L}{a}\left(-\int_{-\infty}^{B^-} + \int_{B^+}^{\infty}\right) 2 \exp\left\{-\left|\frac{\pi(\sin k_j - \Lambda)}{2u}\right|\right\} \sigma_t(\Lambda). \tag{4.7}$$

As in the limit (1.2) $B^\pm = \pm B + O(1/N)$ (this can bee seen comparing(3.8b) and (3.26b)) and $\sigma_t = \sigma_r + O(1/N)$, we may replace $B^\pm$ by $\pm B$, resp. σ_t by σ_r (the errors introduced this way are proportinal to a/u resp. a). Thus, after some simple manipulations we have

$$N\left(\int_{-\infty}^{B^-} + \int_{B^+}^{\infty}\right) 2 \tan^{-1} \tanh \frac{\pi(\sin k_j - \Lambda)}{4u} \sigma_t(\Lambda) \sim$$

$$L\frac{4}{a} \exp\left\{-\frac{\pi B}{2u}\right\} \left(\int_0^{\infty} \exp\left\{-\frac{\pi\Lambda}{2u}\right\} \sigma_r(B + \Lambda)\right) \sinh\left\{\frac{\pi \sin k_j}{2u}\right\}. \tag{4.8}$$

Finally using that ([14], [11])

$$\int_0^{\infty} \exp\left\{-\frac{\pi\Lambda}{2u}\right\} \sigma_r(B + \Lambda) = \frac{2u}{\pi}\sqrt{\frac{\pi}{e}}\sigma(B) + \frac{4u^2}{\pi^2}\sqrt{\frac{\pi}{2e}} \lim_{u\to 0} \sigma_0'(B) \tag{4.9}$$

and

$$\lim_{u\to 0} \sigma(B) = \frac{1}{\sqrt{2}} \lim_{u\to 0} \sigma_0(B) \tag{4.10}$$

to leading order, after substituting the value of B (3.14), we find that the contribution of this term to the r.h.s. of (2.4) is

$$Lm_0 \sinh\left\{\frac{\pi \sin k_j}{2u}\right\}, \tag{4.11}$$

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with m_0 given by (1.2).

The next two terms (line (3.34c), the contribution of the λs and ks) survives the limit. The contribution of the holes and particles in the Λ distribution (line (3.34d) tends to a constant

$$\frac{\pi}{2}(\Delta n^+ - \Delta n^-),\qquad(4.12)$$

while the last two terms (the finite size effects, line (3.34e)) disappear.

In the parity prescription for I_j there is a parameter, $n(\Lambda)$, tending to infinity in the scaling limit. It can be replaced by finite numbers in the following way: first of all, as $n(\Lambda) = n_F + n(p) - n(h)$ with n_F being the number of elements in the Fermi sea, and that the total number of 'excitations' $n(k) + n(p) + n(h) =$even, $n(\Lambda) = n_F + n(k)\,(\mathrm{mod}\,2)$. It is to be noted, that $n_F = N_r + \delta$, i.e. it can be both even and odd, but due to the redefinition of the lattice N_r is always even, thus we have

$$I_j = \frac{n(\lambda) + n(k) + \delta}{2}\,(\mathrm{mod}\,1).\qquad(4.13)$$

Collecting all the terms, and introducing the notations

$$\kappa_j = \frac{\pi \sin k_j}{2u},\quad (n(\kappa) = n(k)),\qquad \chi_\alpha = \frac{\pi \lambda_\alpha}{2u},\quad (n(\chi) = n(\lambda)),\qquad(4.14)$$

we conclude, that the massive particles are described by the equations

$$Lp(\kappa_j) = 2\pi I_j' - \sum_{j'} \phi\left(\frac{\kappa_j - \kappa_{j'}}{\pi}\right) + \sum_\alpha 2\tan^{-1}\left(\frac{\kappa_j - \chi_\alpha}{\pi/2}\right),\qquad(4.15a)$$

$$\left(I_j' = I_j - \frac{\Delta n^+ - \Delta n^-}{4} = \right.$$
$$= \frac{2n(\chi) + 2n(\kappa) + 2\delta - (\Delta n^+ - \Delta n^-)}{4}\,(\mathrm{mod}\,1)$$
$$\left. = \pm\frac{n(\kappa) - 2n(\chi) + 2m}{4} + \frac{n}{2}\,(\mathrm{mod}\,1)\right),$$

$$\sum_j 2\tan^{-1}\left(\frac{\chi_\alpha - \kappa_j}{\pi/2}\right) = 2\pi J_\alpha + \sum_{\alpha'} 2\tan^{-1}\left(\frac{\chi_\alpha - \chi_{\alpha'}}{\pi}\right),\qquad(4.15b)$$

$$\left(J_\alpha = \frac{n(\kappa) + n(\chi) + 1}{2}\,(\mathrm{mod}\,1)\right),$$

with

$$p(\kappa) = m_0 \sinh \kappa\qquad(4.16)$$

Eqs. (4.15) are the secular equations of the massive particles. These equations have the same structure as the corresponding equations of the half-filled band case, with one significant difference: the parity prescription for the quantum numbers I_j' depend not only on the number of excitations, but also on the parameter δ. (For later purposes in the last row of (4.15a) we give the I_j' in terms of other quantum numbers n and m defined in (6.11).)

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V. ENERGY AND MOMENTUM IN THE SCALING LIMIT

1. The energy

Now we calculate the scaling limit of (3.38). First we calculate the contribution of the unbound electrons (lines (3.38d) and (3.38e)). The expression in (3.38d) is exactly the same, as the energy of a massive particle in the half-filled band:

$$-(2t(\cos k - u) - \mu) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi}{2u} \frac{1}{\cosh \frac{\pi(\sin k - \Lambda)}{2u}} = \frac{8t}{\pi} \sqrt{u} \exp\left\{-\frac{\pi}{2u}\right\} ch\kappa \,. \tag{5.1}$$

These terms disappear in the scaling limit. In (3.38e) we may replace $B^{\pm}$ by $\pm B$ and $\varepsilon(\Lambda)$ by $\varepsilon_r(\Lambda)$, and we arrive at

$$\frac{1}{2\pi} \left(\int_{-\infty}^{B^-} + \int_{B^+}^{\infty} \right) \frac{\pi}{2u} \frac{1}{\cosh \frac{\pi(\sin k - \Lambda)}{2u}} \varepsilon(\Lambda) =$$

$$= \frac{1}{u} \exp\left\{-\frac{\pi B}{2u}\right\} \left(\int_0^{\infty} \exp\left\{-\frac{\pi\Lambda}{2u}\right\} \varepsilon_r(B + \Lambda) \right) \cosh\left\{\frac{\pi \sin k}{2u}\right\} \,. \tag{5.2}$$

The integral on the r.h.s. can be evaluated ([14], [11]):

$$\int_0^{\infty} \exp\left\{-\frac{\pi\Lambda}{2u}\right\} \varepsilon_r(B + \Lambda) = \frac{4u^2}{\pi^2} \sqrt{\frac{\pi}{2e}} \lim_{u \to 0} \varepsilon_0'(B) \,. \tag{5.3}$$

(The structure of this relation is the same as that of (4.9), the difference is due to (3.13).) Finally we find that the contribution of the massive particles is simply given by

$$\sum_j \epsilon(\kappa_j) \,, \tag{5.4}$$

where

$$\epsilon(\kappa) = m_0 \cosh \kappa \,. \tag{5.5}$$

Next we consider the energy contribution of the particles and holes in the Λ distribution (3.38c). Due to (3.13)

$$\varepsilon(\Lambda) = \varepsilon_r(\Lambda) + o(B^{\pm} \mp B) \,, \tag{5.6}$$

thus ε can be replaced by ε_r. Moreover, as the ε_r changes on a scale $\propto u$, but $\Lambda^{\pm} \mp B \propto a$, we may linearize ε_r around $\pm B$. Finally we arrive at

$$\sum_p \varepsilon(\Lambda_p) - \sum_h \varepsilon(\Lambda_h) =$$

$$= \sum_p^{(+)} \varepsilon_r'(B)(\Lambda_p^+ - B^+) - \sum_h^{(+)} \varepsilon_r'(B)(\Lambda_h^+ - B^+) -$$

$$- \sum_p^{(-)} \varepsilon_r'(B)(\Lambda_p^- - B^-) + \sum_h^{(-)} \varepsilon_r'(B)(\Lambda_h^- - B^-) +$$

$$+ \varepsilon_r'(B)(\Delta n^+(B^+ - B) + \Delta n^-(-B^- - B)) \,. \tag{5.7}$$

This way the contributions of the particles and holes is expressed by the quantities known from (4.5):

$$\sum_p \varepsilon(\Lambda_p) - \sum_h \varepsilon(\Lambda_h) =$$

$$= \frac{2\pi v}{L}\left\{\sum_p^{(+)}\Delta J_p^+ - \sum_h^{(+)}\Delta J_h^+ + \sum_p^{(-)}\Delta J_p^- - \sum_h^{(-)}\Delta J_h^-\right\} +$$

$$+ 2\pi v\left(\Delta n^+ \frac{\sigma_r(B)(B^+ - B)}{a} + \Delta n^- \frac{\sigma_r(B)(-B^- - B)}{a}\right), \tag{5.8}$$

where

$$v = \frac{a\varepsilon_r'(B)}{2\pi\sigma_r'(B)}. \tag{5.9}$$

We note that due to (3.13), $\varepsilon_r'(\Lambda)$ satisfies an equation of the type (3.12), with $\varepsilon_0(\Lambda)$ replaced by $\varepsilon_0'(\Lambda)$. A consequence of this is that analogously to (4.10)

$$\lim_{u\to 0}\varepsilon_r'(B) = \frac{1}{\sqrt{2}}\lim_{u\to 0}\varepsilon_0'(B), \tag{5.10}$$

which leads to

$$v = 1 \tag{5.11}$$

in the scaling limit.

The terms in line (3.38b), after replacing $\varepsilon'(B^\pm)$ and $\sigma_t(B^\pm)$ by $\varepsilon_r'(B)$ and $\sigma_r(B)$, respectively, turn out to be

$$-\frac{\pi}{6L}. \tag{5.12}$$

Finally we consider the terms in line (3.38a), where the integral is obviously a function of B^- and B^+ but it also depends implicitly on $\varepsilon(\Lambda)$. It can be expanded into a power series of $(B^+ - B)$ and $(B^- + B)$ and after a straightforward calculation we find, that

$$-Nut + N\int_{B^-}^{B^+}\varepsilon(\Lambda)\sigma_0(\Lambda) = E_r + \frac{1}{2}2\pi L\left\{\left(\frac{\sigma_r(B)(B^+ - B)}{a}\right)^2 + \left(\frac{\sigma_r(B)(B^- + B)}{a}\right)^2\right\}. \tag{5.13}$$

Collecting all the terms (5.4-5.13) we have

$$E = E_r - \frac{\pi}{6L} +$$

$$+ \frac{1}{2}2\pi L\left\{\left(\frac{\sigma_r(B)(B^+ - B)}{a}\right)^2 + \left(\frac{\sigma_r(B)(B^- + B)}{a}\right)^2\right\} +$$

$$+ 2\pi\left(\Delta n^+ \frac{\sigma_r(B)(B^+ - B)}{a} + \Delta n^- \frac{\sigma_r(B)(-B^- - B)}{a}\right)$$

$$\frac{2\pi}{L}\left\{\sum_p^{(+)}\Delta J_p^+ - \sum_h^{(+)}\Delta J_h^+ + \sum_p^{(-)}\Delta J_p^- - \sum_h^{(-)}\Delta J_h^-\right\} +$$

$$+ \sum_j \epsilon(\kappa_j). \tag{5.14}$$

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Next one has to calculate the quantities $\sigma_r(B)(B^+ - B)/a$ and $\sigma_r(B)(-B^- - B)/a$ and to do this we use the conditions (3.8b) and (3.26b). Taking the difference of these two equations, and keeping only the terms of order of $1/N$, we arrive at the relation

$$C\left\{\sigma_r(B)(B^+ - B) + \frac{n(p^+) - n(h^+)}{N}\right\} - \sigma_r(B)(B^+ - B) =$$

$$= -\frac{1}{2N}\left(n(\kappa) + \Delta n^+ + \Delta n^- + 2\delta\right), \qquad (5.15)$$

where

$$C = \int_{B}^{\infty} \rho(\Lambda), \qquad (5.16)$$

with ρ determined by the equation

$$\rho(\Lambda) = -\frac{1}{2\pi}K_2(\Lambda - B) - \frac{1}{2\pi}\int_{-B}^{B} K_2(\Lambda - \Lambda')\rho(\Lambda'), \qquad (5.17)$$

and at an analogous relation for $(-B-B^-)$. In the $u \to 0$ limit Eq. (5.17) can be transformed into a Wiener-Hopf type equation and the integral (5.16) can be calculated:

$$C = 1 - \sqrt{2}. \qquad (5.18)$$

This way we find, that

$$\frac{\sigma_r(B)(B^+ - B)}{a} + \frac{n(p^+) - n(h^+)}{L} = -\frac{n(\kappa) + 3\Delta n^+ + \Delta n^- + 2\delta}{2\sqrt{2}L}. \qquad (5.19)$$

Substituting this and the analogous expression for $(-B - B^-)$ into (5.14) yields

$$E = E_r + \sum_j \epsilon(\kappa_j) - \frac{\pi}{6L} +$$

$$+\frac{2\pi}{L}\frac{1}{2}\left\{\left(\frac{n(\kappa) + 2\Delta n^+ + 2\Delta n^- + 2\delta}{2}\right)^2 + \left(\frac{\Delta n^+ - \Delta n^-}{2}\right)^2\right\}$$

$$+\frac{2\pi}{L}\left\{\sum_p^{(+)}\Delta J_p^+ - \sum_h^{(+)}\Delta J_h^+ - \frac{(\Delta n^+)^2}{2}\right\} +$$

$$+\frac{2\pi}{L}\left\{\sum_p^{(-)}\Delta J_p^- - \sum_h^{(-)}\Delta J_h^- - \frac{(\Delta n^-)^2}{2}\right\}. \qquad (5.20)$$

Note, that the values of the expressions in the last two curly brackets are always nonnegative integers.

2. The momentum

The momentum on a lattice is a dimensionless number, while in a continuum it has the dimension $1/length$. In the continuum limit the momentum, P, is obtained by taking the

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limit of P_l/a (with P_l being the lattice momentum). Now P_l is given by (3.40) which after dividing by a and substituting Eqs. (4.15) yields:

$$P = \sum_j p(\kappa_j) +$$
$$+\sum_p^{(+)} \frac{2\pi \Delta J_p^+}{L} - \sum_h^{(+)} \frac{2\pi \Delta J_h^+}{L} -$$
$$-\sum_p^{(-)} \frac{2\pi \Delta J_p^-}{L} + \sum_h^{(-)} \frac{2\pi \Delta J_h^-}{L} +$$
$$+\frac{\pi}{2L}(n(\kappa) + 2\delta)(\Delta n^+ - \Delta n^-), \tag{5.21}$$

that is

$$P = \sum_j p(\kappa_j) + \frac{2\pi}{L} \left(\frac{n(\kappa) + 2\Delta n^+ + 2\Delta n^- + 2\delta}{2} \right) \left(\frac{\Delta n^+ - \Delta n^-}{2} \right) +$$
$$+\frac{2\pi}{L} \left\{ \sum_p^{(+)} \Delta J_p^+ - \sum_h^{(+)} \Delta J_h^+ - \frac{(\Delta n^+)^2}{2} \right\} -$$
$$-\frac{2\pi}{L} \left\{ \sum_p^{(-)} \Delta J_p^- - \sum_h^{(-)} \Delta J_h^- - \frac{(\Delta n^-)^2}{2} \right\}. \tag{5.22}$$

VI. INTERPRETATION

The secular equations, Eqs. (4.5) (for the massless sector) and (4.15) (for the massive sector) do not admit an immediate interpretation in terms of scattering states of a massive and a massless SU(2) doublet as this has been the case for the scaling limit of the HF Hubbard chain [10]. We present below our interpretation of the results obtained in the previous sections.

(i) The ground state (the physical vacuum) is the lowest energy state of N_r bound pairs of bare particles. This state can be excited by intoducing 'particles' and 'holes' into the ground-state distribution of the bound pairs and/or introducing unbound electrons. The former set of excitations resembles a set of massless particles, while the unbound electrons are massive. The two sectors do not completely decouple:

— the total number of (massive and massless) excitations must be even, this is a consequence of the parity prescriptions the quantum numbers must obey, and the requirement, that the Fermi-sea is symmetric;

— the parity prescription for the quantum numbers of the massive particles depends on the state of the massless sector;

— in the energy the numbers of massive and massless particles are nonlinearly coupled.

(ii) As we have noted, the definition of the Fermi sea we used is only one of several (actually infinite) possibilities: the same state can be described in terms of a larger or smaller Fermi sea. We may choose δ even to be a function of of the described state, but the rule

$$n(\kappa) + n(\Lambda) = n_r(\Lambda) + \delta \,(\text{mod}\,2) \tag{6.1}$$

must be obeyed. It is obvious, that $n(p)$ and $n(h)$ (so Δn^+ and Δn^-) depend on the definition of the Fermi sea, but the quantity entering the energy and the momentum

$$n(\kappa) + 2\Delta n^+ + 2\Delta n^- + 2\delta = N_e - 2N_r \tag{6.2}$$

is independent of the definition of δ, as it gives the deviation of the actual bare particle number from that of the reference state. It is easy to see, that the other quantity which enters into the energy and momentum

$$\Delta n^+ - \Delta n^- \tag{6.3}$$

is also independent of δ, hence no physical (measurable) quantity depends on our choice of the Fermi sea. It is, however, unavoidable to make a definite choice in order to be able to define the massless particles.

A simple consequence of the above arbitrariness is that we can formally 'minimize' the coupling between the massive and massless sectors: we can choose a δ (obeying (6.1)) such that $\delta' = n(\kappa)/2 + \delta$ have the smallest modulus. There are four inequivalent values for δ': 0 or 1, if $n(\kappa)$ is even, and $+1/2$ or $-1/2$, if $n(\kappa)$ is odd. We note, that Eq. (6.2) has now the form

$$2\Delta n^{+\prime} + 2\Delta n^{-\prime} + 2\delta' = N_e - 2N_r \,. \tag{6.4}$$

The 'minimizing' of the coupling between the massive and massless sectors corresponds to choosing the Fermi level with $2\Delta n^{+\prime} + 2\Delta n^{-\prime}$ as close to $N_e - 2N_r$ as possible. It is worth to mention that in the HF band case $2N_r - N_e = N - N_e$ is measured by the number of massless particles only, in this sense the above choice of δ' mimicks most the HF case.

(iii) To specify a state it is not sufficient to give the number of particles (and the quantum numbers) in the two sectors, but one also needs the value of δ or δ'. It is not hard to see, that any state with given $n(\kappa)$, $n(p)$ and $n(h)$ can have both $\delta = 0$ and $\delta = 1$ (if we use the convention (3.18)), or equivalently can have both $\delta' = 0$ and $\delta' = 1$ for $n(\kappa)$ even, and $\delta' = \pm 1/2$ for $n(\kappa)$ odd. In this sense δ (or δ') is a free parameter. States differing only in the value of δ (δ') are not degenerate, as δ appears in the parity prescription for I'_j in (4.15a), and it appears explicitly in the energy (5.20) too.

(iv) The massive sector described by Eqs. (4.15) consists of (relativistic) particles in the doublet representation of SU(2). Their contribution to the energy and momentum is not simply the sum of the individual contributions but there are terms both in the energy and momentum, which depend on the *number of massive particles* in a nontrivial way. (As these terms contain data on the massless sector too, we interpret them as a coupling between the two sectors.)

Any solution of Eqs. (4.15) corresponds to a highest weight state of SU(2) with $S^2 = l(l+1)$, $S^z = m$, $l = m = (n(\kappa) - 2n(\chi))/2$. All the other members of the $l = (n(\kappa) - 2n(\chi))/2$ multiplets are degenerate with this state, and can be obtained by the action of the $\sigma^- = \sum_i c_{i,\downarrow}^+ c_{i,\uparrow}$ operator.

The BA Eqs. (4.15) are of the familiar type and this makes possible extract the two particle scattering matrices in the usual way [17–19]. For this we have to choose those solutions of Eqs. (4.15) which correspond to the triplet and singlet states with an *empty massless sector*. To avoid ambiguities due to the freedom in choosing the Fermi level one should first define the state with no massless particles. It is natural to consider the massless sector empty, if its energy contribution is zero. This corresponds to $m = n = 0$ (see the next point), and then the scattering matrix of the massive doublet is given by

$$\hat{S}(\Delta\kappa) = -\exp\left\{ i\phi\left(\frac{\Delta\kappa}{\pi}\right) \right\} \frac{\Delta\kappa\hat{I} - i\pi\hat{\Pi}}{\Delta\kappa - i\pi}, \tag{6.5}$$

where $\hat{I}$ resp. $\hat{\Pi}$ are the identity resp. permutation operators acting on the spins of the two particles:

$$I_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'} = \delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}, \quad \Pi_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'} = \delta_{\sigma_1'\sigma_2}\delta_{\sigma_2'\sigma_1}. \tag{6.6}$$

(v) The massless sector described by Eqs. (4.5). looks very much like as if it consisted of free particles. This picture, however, cannot be taken at face value as the definition of the particles (and holes) is not unique. Also as both the energy and momentum contain the *collective* terms it looks as if there were interactions. One cannot, however, extract phaseshifts from Eqs. (4.5).

The energy and momentum can be described in terms of a CFT. If the massive sector is empty the energy momentum dispersion shows a tower structure: the central charge is $c = 1$, and the apexes of the towers are located at

$$E - E_r = -\frac{\pi}{6L} + \frac{2\pi}{L} x_{n,m}, \quad P = \frac{2\pi}{L} s_{n,m}, \tag{6.7}$$

with

$$x_{n,m} = \frac{1}{2}\left(n^2 + m^2\right), \quad s_{n,m} = nm, \tag{6.8}$$

where the integers n and m are

$$n = \left(\Delta n^+ + \Delta n^- + \delta\right), \quad m = \left(\frac{\Delta n^+ - \Delta n^-}{2}\right). \tag{6.9}$$

This leads to the conformal weights (the notations are those of [20])

$$\Delta = \frac{1}{4}(n+m)^2, \quad \bar{\Delta} = \frac{1}{4}(n-m)^2. \tag{6.10}$$

(We have to note, that even if the massive sector is not empty, the contribution of the massless sector is of the tower structure, just with

$$n = \left(\frac{n(\kappa) + 2\Delta n^+ + 2\Delta n^- + 2\delta}{2}\right), \quad m = \left(\frac{\Delta n^+ - \Delta n^-}{2}\right). \tag{6.11}$$

In this case, however, due to the coupling of the massive sector, the apexes of the towers can not be interpreted as anomalous dimensions and spins. For $n(\kappa) \neq 0$ n and m can be half-integers too: $2n, 2m = n(\kappa)\,(\mathrm{mod}\,1)$.) The conformal weights (6.10) correspond to a SU(2)×SU(2) symmetric CFT. It should be interesting to directly identify the corresponding SU(2)×SU(2) multiplet structure.

(vi) The spectrum of the limiting model is the same both in the NHF and the HF band case. Actually the complete energy and momentum are

$$E = E_r + \sum_j \epsilon(\kappa_j) - \frac{\pi}{6L} + \frac{2\pi}{L}\left(\frac{n^2 + m^2}{2}\right) + \frac{2\pi}{L}\left(\nu^+ + \nu^-\right), \tag{6.12}$$

$$P = \sum_j p(\kappa_j) + \frac{2\pi}{L}nm + \frac{2\pi}{L}\left(\nu^+ - \nu^-\right). \tag{6.13}$$

The towers are obviously the same as in the HF case, and so is the contribution of the massive sector, as the parity prescription for the quantum numbers in (4.15) (i.e. the quantization of the momenta of the massive particles) when expressed in terms of n and m agrees with that of the HF case. We note, however, that this way we have only shown that the points of the two spectra coincide, but we can not say anything about the degeneracies of the single points.

Finally let us briefly present the results for the ground state energy of a finite density state (i.e. when the number of excitations is macroscopic) of the limiting model. As δ can be choosen to minimize the coupling between the sectors, one can discuss the massive and massless sectors separately. (The coupling due to the parity-prescription for I'_j and the explicit presence of δ' in the energy of the massless sector may actually give a contribution $O(1/L)$ in the energy density.) Since then the massive sector is described by the same equations and energy-momentum dispersion as in the HF-band case, one can literally take over the results obtained there [10] and we do not reproduce the formulae here.

The treatment of the massless sector is extremely simple. After choosing the δ' as described in the previous section, the energy density associated with the massless sector is just given as

$$\bar{\mathcal{E}}(\varrho) = \pi\varrho^2, \tag{6.14}$$

where ϱ is the density of massless particles. Eq. (6.14) coincides with the energy density of noninteracting particles.

The total free energy density (defined as the Legendre transformation of $\mathcal{E}(\varrho)$ with a chemical potential ν) will be simply the sum of the massive and massless contributions just as in the HF case. The decoupling of the two sectors depends apparently on our choice of the Fermi level. We can understand this as follows. There are two energy contributions controled by two independent parameters. One is the energy of the massive particles, related to their relativistic dispersion, this is determined by $n(\kappa)$. The other contribution is a (quadratic) function of $N_e - 2N_r$, i.e. the deviation of the actual particle number from that of the reference state. If we make the Legendre transformation in this two variables, we shall get two independent terms. If we choose some combinations of $n(\kappa)$ and $N_e - 2N_r$ as independent variables, the free energy will have terms depending on both chemical potentials. (In the HF-band case $2N_r - N_e = N - N_e$ is measured by the number of massless particles, thus the separation of the two sectors is obvious. For the NHF-band case measuring $N_e - 2N_r$ can be done in different ways. After having choosen the δ', $N_e - 2N_r \sim 2L\varrho$. The factor 2 in this equation appears as ϱ is the density of bound *pairs* and this is the reason for the discrepancy of (6.14) from the analogous expression in the HF case.)

In conlusion the above result provide strong (although indirect) evidence that the scaling limit of the NHF Hubbard chain is the SO(4) symmetric CGN model.

VII. DIFFERENCES BETWEEN THE HF AND NHF CASES

In the formulae (1.2) one can take the $n \to 1$ limit suggesting that there is a smooth limit from the NHF to the HF case. In this section we list some differences between the HF and NHF cases, which clearly show, that in spite of the 'continuity' of (1.2) in n the scaling limit of the NHF and that of the HF case can not be related in a trivial way (in other words the $n \to 1$ and the scaling limit do not commute in a obvious way).

- In the HF case the ground state is well defined, and in the description of the bound pairs (Λ) the Fermi surface plays no role: the Fermi points in the rapidity space ($B^{\pm}$) are at infinity. As a consequence, there are no particles, only holes connected with the Λ distribution. These elementary excitations (we call them massless particles) carry an SU(2) isospin and the states are isospin eigenstates often characterized by a set of complex variables. In the NHF case the Fermi surface is at some finite rapidity, there are both particles and holes, but in the limiting theory there are no states with complex Λs, as those require an infinite excitation energy.

- In the HF case the two kinds of excitations have two different internal degrees of freedom (spin resp. isospin), and the ground state is a singlet of both. In the NHF case one of the excitations carries the spin, but the isospin of the state cannot be uniquely attached to one or the other kind of excitation. The reference state is a spin singlet, but its isospin $(N - 2N_r)/2 \to \infty$ in the scaling limit. (The difference between the isospin of an excited state and that of the reference state is, however, finite: $\Delta I_3 = -(n(\kappa) + 2\Delta n^+ + 2\Delta n^- + 2\delta)/2)$

- In the HF chain the massless particles are uniquely defined, in the NHF case — although the state can be given uniquely — the definition of the particles and holes connected with the Λ distribution (massless particles) is not unique.

- In both cases the conformal dimensions of the massless sector correspond to an SU(2)×SU(2) symmetric CFT. In the case of the HF chain an SU(2) symmetry corresponding to the gapless excitations is already present in the lattice model, and this with the separation of the left and right sectors develops into an SU(2)×SU(2) symmetry. For the NHF chain in the lattice model there is no such SU(2) symmetry, which could be identified in the massless sector and the SU(2)×SU(2) symmetry (whose existence is indicated by the conformal weights) appears only in the scaling limit.

- While for the HF case the conformal weights Δ and $\bar{\Delta}$ are connected to the right resp. left going massless particles, for the NHF case both Δ and $\bar{\Delta}$ depend on both the right $(+)$ and left $(-)$ part of the massless sector.

APPENDIX A:

We sketch here briefly the 'naive' continuum limit of the less than half filled Hubbard chain. A more detailed description of the procedure is given in Ref. [21].

First we write the (1.1) Hamiltonian in the form

$$
\hat{H} = -t \sum_{i=1}^{N} \sum_{\sigma=\uparrow,\downarrow} \left(c_{i,\sigma}^{+} c_{i+1,\sigma} + h.c. \right) + \frac{U}{2} \sum_{i=1}^{N} (\hat{n}_{i,\uparrow} + \hat{n}_{i,\downarrow})^2
$$
$$
+ h \sum_{i=1}^{N} (\hat{n}_{i,\uparrow} + \hat{n}_{i,\downarrow}) + \frac{NU}{4} , \tag{A1}
$$

with $h = \mu - U$.

We have seen earlier, that in order to avoid divergent terms in the momentum we have to redefine the lattice so, that in the new lattice ν sites form one elementary cell. (ν together with an other number η is defined so, that if the bandfilling is $n = l/q$, where l and q are relative prime numbers, than ν and η are the smallest integers satifying $\nu = 4q\eta/l$.) We define the operators

$$
\phi_{\alpha,\sigma}(n) = \frac{1}{\nu\sqrt{a}} \sum_{j=1}^{\nu} e^{-ik\alpha j} c_{\nu(n-1)+j,\sigma} \quad \alpha = 1, 2, \ldots, \nu, \quad \sigma =\uparrow,\downarrow, \quad k = \frac{2\pi}{\nu}. \tag{A2}
$$

In terms of these the original Fermion operators are

$$
c_{\nu(n-1)+j,\sigma} = \sqrt{a} \sum_{\alpha=1}^{\nu} e^{ik\alpha j} \phi_{\alpha,\sigma}(n) \quad j = 1, 2, \ldots, \nu. \tag{A3}
$$

The part of the Hamiltonian quadratic in the field operators now reads

$$
\sum_{n=1}^{N'} \sum_{\sigma} \sum_{\alpha=1}^{\nu} \nu a(h - 2t \cos k\alpha)\phi_{\alpha,\sigma}^{+}(n)\phi_{\alpha,\sigma}(n)
$$
$$
-at \sum_{n=1}^{N'} \sum_{\sigma} \sum_{\alpha,\beta=1}^{\nu} \phi_{\alpha,\sigma}^{+}(n) \left(\phi_{\beta,\sigma}(n+1) - \phi_{\beta,\sigma}(n) \right) e^{ik\beta}
$$
$$
+at \sum_{n=1}^{N'} \sum_{\sigma} \sum_{\alpha,\beta=1}^{\nu} \phi_{\alpha,\sigma}^{+}(n) \left(\phi_{\beta,\sigma}(n) - \phi_{\beta,\sigma}(n-1) \right) e^{-ik\alpha}, \tag{A4}
$$

with $N' = N/\nu$ being the number of new elementary cells. This expression shows that the procedure leads to a meaningful result only, if for some $\alpha = \alpha_0$ and $\alpha = \nu - \alpha_0$

$$h - 2t\cos k\alpha = 0, \tag{A5}$$

otherwise all the new fields become infinitely massive if $t \to \infty$. Obviously

$$\frac{2\pi\alpha_0}{\nu} = k_F(= \frac{\pi n}{2}) \tag{A6}$$

what implies, that

$$\alpha_0 = \eta, \tag{A7}$$

with η defined above. It is clear, that all the modes $\alpha \neq \eta, \nu - \eta$ can be omitted, as they become infinitely massive in the continuum limit.

The length of the chain is

$$L = Na = \nu a N' \; (N' = int.) \tag{A8}$$

and the continuum limit is defined as

$$a \to 0, \quad N' \to \infty, \quad L = fixed$$

with the continuous variable

$$x = \nu a(n - 1/2), \quad dx = \nu a. \tag{A9}$$

In this limit

$$\sum_n^{N'} \to \int_0^L \frac{dx}{\nu a} \quad \text{and} \quad \delta_{n,n'} \to \nu a \delta(x - x'). \tag{A10}$$

If we introduce

$$\phi_{\eta,\sigma}(n) = \psi_{1,\sigma}(x), \quad \phi_{\nu-\eta,\sigma}(n) = \psi_{2,\sigma}(x), \tag{A11}$$

then

$$\left\{\psi_{\alpha,\sigma}(x), \psi_{\beta,\sigma'}^+(x')\right\} = \delta(x - x')\delta_{\alpha,\beta}\delta_{\sigma,\sigma'}. \tag{A12}$$

Applying (A3), (A11) and (A10) to the Hubbard Hamiltonian (A1) one obtains in the naive continuum limit the following Hamiltonian density:

$$\mathcal{H}(x) = (2at\sin\pi n/2)\left(-\sum_{\sigma=1}^{2}\psi_\sigma^+\gamma_5\partial_x\psi_\sigma+ \right. \tag{A13}$$

$$\left. + u\left[\sum_{\sigma\sigma'}\left(\psi_{1,\sigma}^+\psi_{2,\sigma}\psi_{2,\sigma'}^+\psi_{1,\sigma'} + \psi_{2,\sigma}^+\psi_{1,\sigma}\psi_{1,\sigma'}^+\psi_{2,\sigma'}\right) + \left(\sum_\sigma\psi_\sigma^+\psi_\sigma\right)^2\right]\right) \tag{A14}$$

where

$$\psi_\sigma = \begin{pmatrix}\psi_{1,\sigma}\\ \psi_{2,\sigma}\end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \tag{A15}$$

and $u = U/4t\sin\pi n/2$. (Note that this differs from the definition used in the bulk of the paper as there $u = |U|/4t$.)

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